SYNTHESIS OF TWO-DIMENSIONAL LOSSLESS m -PORTS WITH PRESCRIBED SCATTERING MATRIX*

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Abstract. Multidimensional lossless networks are of special interest for use as reference structures for multidimensional wave digital filters [1]–[3]. The starting point of the presented synthesis procedure for two-dimensional representatives of the networks mentioned is a scattering matrix description of the desired multiport. This given matrix is assumed to have those properties which have turned out to be necessary [9], [10] for any scattering matrix of a multidimensional lossless network. The method presented for the synthesis of 2-D reactance m -ports is based mainly on known properties of block-companion matrices and the factorization of a univariable rational matrix which is discrete para-Hermitian and nonnegative definite on the unit circle.

The resulting network always contains only a minimal number of frequency-dependent building elements. No restrictions are made concerning the coefficients of the rational entries of the scattering matrix; they may be either real or complex, so as to include even complex networks which are of special interest for multidimensional wave digital filters [3].

1. Introduction

The theory of multidimensional networks has been drawing attention for several years, and attained new importance in the context of multidimensional wave digital filters (WDF), since the main problem is that of finding a suitable reference structure, whereas the design of the corresponding WDF is rather straightforward.

In the late 1960s [12] the synthesis of two-variable lossless multiports had already been considered, but certain properties of reactance Hurwitz polynomials [13] that are needed in the context of this theory, were not then available, so that the proof formerly given cannot be considered to be

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fully complete. In the context of the synthesis of networks containing lumped and distributed elements [11], scattering matrices have been used, as in this paper. However, the algorithm in [11] is very complicated because it involves a Taylor expansion of a two-variable rational matrix which will cause convergence problems which have not been discussed in [11]. The occurrence of singularities of the second kind on the distinguished boundary as well as the extension to complex systems also have not been treated. A detailed discussion of [11] and [12] can be found in the introduction and the appendix of [7]. A further attempt to find a general synthesis method for this class of networks has turned out to be incorrect (see [5] together with Section 5.3.2 of [6]).

The basic idea of the synthesis procedure presented for 2-D lossless networks with lumped elements has already been introduced in a shortened version in a conference paper [4]. Some involved proofs given there are now simplified and others, which had to be omitted because of lack of space, are now outlined in detail.

It seems to suggest itself to perform the mentioned synthesis in the $\mathbf{p}$ -domain, however, it turned out that it is appropriate to use a $\mathbf{z}$ -domain approach, where the complex variables p_i and z_i , $i = 1, 2$, are related by the well-known bilinear transformation (4). Hence, we do not make use of matrices obeying the conditions established in [9] and [10] but use so-called *discrete lossless bounded matrices* (DLBM) [6], [14] which are the $\mathbf{z}$ -domain counterparts to rational bounded paraunitary matrices in the $\mathbf{p}$ -domain.

After giving some preliminary definitions in Section 2, the actual synthesis procedure is outlined in Section 3 which results in a minimal solution to the problem mentioned. In the Appendix, some of the mathematical tools related to Section 3 are presented.

2. Definitions and notations

We consider rational functions or matrices which are expressions in $k \geq 1$ complex variables, z_i , $i = 1$ to k . For convenience of notation we summarize them in a vector

$$\mathbf{z} = (z_1, \dots, z_k).$$

A notation such as $|\mathbf{z}| < 1$ designating $|z_i| < 1$ for all $i = 1$ to k . Obvious variations of this notation with the symbol $<$ replaced by $=$, $\leq$, etc., are used.

If a polynomial matrix $\mathbf{P}(\mathbf{z})$ (or a polynomial in the scalar case) is written as a polynomial matrix in the variable z_i , $i \in \{1, \dots, k\}$, as

$$\mathbf{P} = \sum_{\nu=0}^{n_i} \mathbf{P}_{\nu} z_i^{\nu},$$

where the coefficients $\mathbf{P}_{\nu}$ are polynomial matrices in the remaining variables with $\mathbf{P}_{n_i} \neq \mathbf{0}$, then $n_i = \deg_i \mathbf{P}$ is the partial degree of $\mathbf{P}$ in z_i .

The *Hermitian conjugate* (complex conjugation + transposition), $\mathbf{S}^*$, of a matrix $\mathbf{S}$ is designated by a superscript asterisk.

The *discrete paraconjugate*, $\tilde{\mathbf{S}}(\mathbf{z})$, of a given rational matrix $\mathbf{S}$ is defined as

$$\tilde{\mathbf{S}} = \mathbf{S}^*(z_1^{*-1}, z_2^{*-1}, \dots, z_k^{*-1}).$$

For $|\mathbf{z}| = 1$ we have

$$\tilde{\mathbf{S}}(\mathbf{z}) = \mathbf{S}^*(\mathbf{z}). \quad (1)$$

The *reciprocal conjugate*, $\hat{g}$, of a polynomial $g = g(z_1, \dots, z_k)$ in k variables $\mathbf{z}$ is defined as

$$\hat{g} = \tilde{g} z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k},$$

where $n_i = \deg_i g$, $i = 1$ to k .

We use the following notations:

$$U^k = \{(z_1, \dots, z_k) \mid |z_i| < 1, \forall i \in \{1, \dots, k\}\}$$

is the open unit polydisk,

$$\bar{U}^k = \{(z_1, \dots, z_k) \mid |z_i| \leq 1, \forall i \in \{1, \dots, k\}\}$$

is the closed unit polydisk, and

$$T^k = \{(z_1, \dots, z_k) \mid |z_i| = 1, \forall i \in \{1, \dots, k\}\}$$

is the distinguished boundary of the unit polydisk.

Next, we recall the definition of a scattering Schur polynomial [14].

Definition 1. A polynomial $g = g(\mathbf{z})$ is called *scattering Schur* (SS) if $g \neq 0$ for $|\mathbf{z}| < 1$, and g and $\hat{g}$ do not have any common proper factor.

Definition 2. An $(m \times m)$ -matrix $\mathbf{S}(\mathbf{z})$ is said to be a *discrete lossless bounded matrix* (DLBM) if

- (i) $\mathbf{S}(\mathbf{z})$ is rational in $\mathbf{z}$,
- (ii) the entries S_{ij} of $\mathbf{S}$ are unimodularly bounded in U , i.e.,

$$|S_{ij}| \leq 1 \quad \text{for } |\mathbf{z}| < 1,$$

- (iii) $\mathbf{S}\tilde{\mathbf{S}} = \tilde{\mathbf{S}}\mathbf{S} = \mathbf{1}_m$,

where $\mathbf{1}_m$ is the unit matrix of order m .

A rational matrix, $\mathbf{S}$, is *regular* if its determinant, $\det \mathbf{S}$, is not identically zero. $\mathbf{S}$ has the *normal rank*, $r = \text{rank } \mathbf{S}$, if at least one $(r \times r)$ -minor of $\mathbf{S}$ is not identically zero, and all $(l \times l)$ -minors of $\mathbf{S}$, with $l > r$, are identically zero. Whereas the local rank, $\text{loc-rank } \mathbf{S}(\mathbf{z}_0)$, designates the rank of $\mathbf{S}$ in the normal sense for the fixed point $\mathbf{z}_0$. Notice that

$$\text{loc-rank } \mathbf{S}(\mathbf{z}_0) \leq \text{rank } \mathbf{S}.$$

A polynomial matrix is called *unimodular* if its determinant is a constant with modulus one.

3. Synthesis procedure

Before the actual synthesis method can be developed, we have to consider some preliminary results.

Theorem 1. *An $(m \times m)$ -matrix $S(z)$ is a DLBM if*

- (i) $S(z)$ is rational in z ,
- (ii) $S(z)$ is holomorphic for $|z| < 1$,
- (iii) $S\tilde{S} = \tilde{S}S = 1_m$.

Proof. From (iii) we have

$$\sum_{j=1}^m S_{ij} \tilde{S}_{ij} = 1, \quad i = 1 \text{ to } m, \quad (2)$$

where the S_{ij} 's are the entries of S . Thus, it follows from (1) and (2)

$$\sum_{j=1}^m |S_{ij}|^2 = 1, \quad i = 1 \text{ to } m, \quad \text{for } z \in T^k,$$

i.e., $|S_{ij}| \leq 1$ for all those points $z \in T^k$ for which S_{ij} has no nonessential singularity of the second kind. Furthermore, due to (ii), S_{ij} is holomorphic for $|z| < 1$. These two properties, together with the main theorem (version A) of [8], lead to the desired result:

$$|S_{ij}| \leq 1 \quad \text{for } |z| < 1. \quad (3)$$

The proof is completed by comparing (3), (i), and (iii) with Definition 2. $\square$

As already mentioned, we do not solve our problem in the $\mathbf{p}$ -domain but make use of the bilinear transformation

$$z_i = \frac{1-p_i}{1+p_i}, \quad p_i = \frac{1-z_i}{1+z_i}, \quad i = 1 \text{ to } k, \quad (4)$$

which maps the open right half-polyplane of the $\mathbf{p}$ -domain into U^k of the z -domain. Thus, as illustrated in Figure 1, the reflectance of a p_i -type unit-capacitance is simply z_i . The advantage of the transformation (4) results

Figure 1. The reflectance of a unit-capacitance.

from the fact that, when dealing with the scattering matrix, it is appropriate to consider the reflectance of the frequency-dependent one-ports of the corresponding network instead of their imittances. Therefore, we use the terminology “ z_i -type reflectance” instead of “ p_i -type unit-capacitance,” i.e., we make use of the correspondence shown in Figure 1. From transformation (4) and the necessary properties of the scattering matrix of a lossless network with lumped elements, established in [9] and [10], we obviously obtain the following corollary.

Corollary 1. *The scattering matrix in the z -domain (called discrete scattering matrix), $S(z)$, of a lossless network with lumped elements is a DLBM (see Definition 2).*

Conversely, is it possible to realize every DLBM as the discrete scattering matrix of a lossless multiport with lumped elements? From classical network theory it is known that this question can be answered with yes for constant and one-variable DLBMs, but what about multivariable DLBMs?

Subsequently, it will be shown that every $(m \times m)$ two-variable DLBM, $S(z_1, z_2)$, can be realized with a minimal number of frequency-dependent elements. This synthesis procedure can be divided into two steps. First we try to find an $(m + mn_2) \times (m + mn_2)$ one-variable discrete scattering matrix, $S'(z_1)$, which describes a z_1 -dependent one-dimensional $(m + mn_2)$ -port, N' , where n_2 will be specified later. The matrix $S'(z_1)$ must not be a DLBM, we only demand that we obtain a two-dimensional m -port, N , which is described by the given DLBM $S(z)$, if the last mn_2 ports of N' are terminated with z_2 -type reflectances (see Figure 2). The signals at the z_2 -dependent one-ports can be interpreted as internal states. Thus, it is possible to apply a similarity transformation to S' in such a way that the internal structure of N is changed without influencing the external behavior, which is described by S .

In the second step we use such a similarity transformation in order to obtain a new one-variable matrix, $S''(z_1)$, describing a coupling network, N'' , which can be realized with known methods, i.e., S'' must have some special features, especially, a certain block-element of S'' is a DLBM.

3.1. First synthesis step

Given a two-variable $(m \times m)$ DLBM, $S(z_1, z_2)$, we try to find a rational matrix $S'(z_1)$ of order $(m + mn_2)$ which satisfies the condition already mentioned, which is also illustrated in Figure 2. Simple matrix manipulation shows that the requirement

$$S = H_{11} + H_{12}(z_2^{-1} \mathbf{1}_{mn_2} - H_{22})^{-1} H_{21}, \quad (5)$$

Figure 2. Synthesis of a two-dimensional lossless m -port.

where

$$\begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix} \begin{matrix} m \\ mn_2 \end{matrix} = \mathbf{S}', \quad (6)$$

is sufficient to fulfill the above-stated condition. Now we have to determine the block-elements of $\mathbf{S}'$ in terms of the elements of $\mathbf{S}$ in such a way that equation (5) holds.

To do this we recall two properties of the DLBM $\mathbf{S}(\mathbf{z})$ [14, Property 1]:

$$(i) \quad \det \mathbf{S} = \sigma \frac{\hat{g}}{\hat{g}}, \quad (7)$$

where $\hat{g}$ is an SS polynomial (see Definition 1) and σ is a constant such that $|\sigma| = 1$.

(ii) $\mathbf{S}$ can be written as

$$\mathbf{S} = \frac{\mathbf{P}}{\hat{g}}, \quad (8)$$

where $\mathbf{P}$ is a polynomial matrix and g is as in (i) above.

Furthermore, the determinant of the $(m \times m)$ -matrix $\mathbf{S}$ can be expressed as $\det \mathbf{S} = (\det \mathbf{P}) / \hat{g}^m = \sigma(g / \hat{g})$, i.e., $\det \mathbf{P} = \sigma g \hat{g}^{m-1}$, i.e.,

$$m \deg_i \mathbf{P} \geq \deg_i \det \mathbf{P} = \deg_i g + (m-1) \deg_i \hat{g} \geq m \deg_i \hat{g},$$

since from the definition of the reciprocal conjugate of a polynomial we can conclude

$$\deg_i g \geq \deg_i \hat{g}. \quad (9)$$

Thus, we have

$$n_i = \deg_i \mathbf{P} \geq \deg_i \hat{g}, \quad i = 1, 2. \quad (10)$$

In the following we assume

$$n_i \geq 1, \quad i = 1, 2, \quad (11)$$

since otherwise $\mathbf{S}(z)$ would not actually depend on two variables.

Inequalities (10) and (11) imply that $\mathbf{P}$ and $\hat{g}$ can be written as

$$\mathbf{P}(z) = \mathbf{P}_0(z_1) + \mathbf{P}_1(z_1)z_2 + \cdots + \mathbf{P}_{n_2}(z_1)z_2^{n_2}, \quad (12a)$$

$$\hat{g}(z) = a_0(z_1) + a_1(z_1)z_2 + \cdots + a_{n_2}(z_1)z_2^{n_2}, \quad (12b)$$

where the coefficients $\mathbf{P}_i(z_1)$ and $a_i(z_1)$, $i = 0$ to n_2 , are one-variable polynomial matrices and polynomials, respectively, with $\mathbf{P}_{n_2}(z_1) \neq \mathbf{0}$. Besides this, since $\hat{g}$ is SS, the one-variable polynomial $a_0(z_1) = \hat{g}(z_1, 0)$ is SS too (see Theorem 11 of [14]), i.e.,

$$a_0(z_1) \neq 0 \quad \text{in} \quad |z_1| < 1. \quad (13)$$

For convenience of notation we define

$$u_i = \frac{a_i}{a_0}, \quad i = 0 \text{ to } n_2, \quad (14)$$

$$\mathbf{N}_i = \frac{\mathbf{P}_i}{a_0}, \quad i = 0 \text{ to } n_2, \quad (15)$$

$$\mathbf{A}_i = \mathbf{N}_i - \mathbf{N}_0 u_i, \quad i = 1 \text{ to } n_2. \quad (16)$$

Theorem 2. *The following defined matrices $\mathbf{H}_{11}$, $\mathbf{H}_{12}$, $\mathbf{H}_{21}$, and $\mathbf{H}_{22}$ are a solution of equation (5):*

$$\mathbf{H}_{11} = \mathbf{N}_0, \quad (17a)$$

$$\mathbf{H}_{21} = [\mathbf{A}_{n_2}^T \quad \mathbf{A}_{n_2-1}^T \quad \cdots \quad \mathbf{A}_2^T \quad \mathbf{A}_1^T]^T, \quad (17b)$$

$$\mathbf{H}_{12} = [\mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{1}_m], \quad (17c)$$

$$\mathbf{H}_{22} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -u_{n_2}\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0} & \cdots & \mathbf{0} & -u_{n_2-1}\mathbf{1}_m \\ \mathbf{0} & \mathbf{1}_m & \cdots & \mathbf{0} & -u_{n_2-2}\mathbf{1}_m \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_m & -u_1\mathbf{1}_m \end{bmatrix}, \quad (17d)$$

where

$$\mathbf{H}_{12} = \mathbf{1}_m \quad \text{and} \quad \mathbf{H}_{22} = -u_1\mathbf{1}_m \quad \text{for} \quad n_2 = 1.$$

Proof. Because the proof is trivial for $n_2 = 1$, we restrict further considerations to the case $n_2 > 1$. First we calculate the matrix $(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})$ by using (17d), and partition it in blocks:

$$\begin{aligned} (z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22}) &= \left[\begin{array}{ccccc|c} z_2^{-1}\mathbf{1}_m & \mathbf{0} & \cdots & \cdots & \mathbf{0} & u_{n_2}\mathbf{1}_m \\ -\mathbf{1}_m & z_2^{-1}\mathbf{1}_m & \cdots & \cdots & \mathbf{0} & u_{n_2-1}\mathbf{1}_m \\ \mathbf{0} & -\mathbf{1}_m & \ddots & & \mathbf{0} & u_{n_2-2}\mathbf{1}_m \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & -\mathbf{1}_m & z_2^{-1}\mathbf{1}_m & u_2\mathbf{1}_m \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{1}_m & (z_2^{-1} + u_1)\mathbf{1}_m \end{array} \right] \\ &= \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \begin{matrix} (n_2-1)m \\ m \end{matrix} \\ &\quad \begin{matrix} (n_2-1)m & m \end{matrix} \end{aligned} \quad (18)$$

Now let us define the matrix

$$\mathbf{D} = (\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}), \quad (19)$$

where

$$\mathbf{M}_{11}^{-1} = \begin{bmatrix} z_2\mathbf{1}_m & \mathbf{0} & \cdots & \mathbf{0} \\ z_2^2\mathbf{1}_m & z_2\mathbf{1}_m & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ z_2^{n_2-1}\mathbf{1}_m & z_2^{n_2-2}\mathbf{1}_m & \cdots & z_2\mathbf{1}_m \end{bmatrix}. \quad (20)$$

If we take into account (18), (20), and (19), we obtain

$$\mathbf{D} = (u_{n_2}z_2^{n_2-1} + u_{n_2-1}z_2^{n_2-2} + \cdots + u_2z_2^1 + u_1 + z_2^{-1})\mathbf{1}_m,$$

and with (14) and (12b)

$$\mathbf{D} = \frac{\hat{g}(\mathbf{z})}{z_2 a_0(z_1)} \mathbf{1}_m. \quad (21)$$

Since the matrices $\mathbf{M}_{11}$ (see (20)) and $\mathbf{D}$ are obviously invertible, we can apply A1 (see the Appendix) to $(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})$. To do this we use the block-representation (18), A1, and the definition of $\mathbf{D}$ (19):

$$(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})^{-1} = \left[\begin{array}{c|c} \cdots & \cdots \\ \hline -\mathbf{D}^{-1}\mathbf{M}_{21}\mathbf{M}_{11}^{-1} & \mathbf{D}^{-1} \end{array} \right].$$

Now, we put (20) and (18) into the above equation, and use the definitions (17b) and (17c):

$$\mathbf{H}_{12}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})^{-1}\mathbf{H}_{21} = \mathbf{D}^{-1}(\mathbf{A}_{n_2}z_2^{n_2-1} + \mathbf{A}_{n_2-1}z_2^{n_2-2} + \cdots + \mathbf{A}_2z_2 + \mathbf{A}_1). \quad (22)$$

In the last step, we put (21), (16), (15), and (14) into equation (22) and obtain, by an appropriate transformation of the new expression and by comparing this result with (12),

$$\mathbf{H}_{12}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})^{-1}\mathbf{H}_{21} = \frac{\mathbf{P}}{\hat{g}} - \mathbf{N}_0,$$

and with (17a) and (8) we have

$$\mathbf{S} = \frac{\mathbf{P}}{\hat{g}} = \mathbf{H}_{11} + \mathbf{H}_{12}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})^{-1}\mathbf{H}_{21}. \quad (23)$$

The proof is completed by comparing (23) with (5). $\square$

The matrix

$$\mathbf{S}' = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix}, \quad (24)$$

where $\mathbf{H}_{11}$, $\mathbf{H}_{12}$, $\mathbf{H}_{21}$, $\mathbf{H}_{22}$ are defined by (17), is rational in z_1 and a solution of equation (5). Furthermore, it is holomorphic in $|z_1| < 1$, since the denominator polynomial of $\mathbf{S}'$ is composed only of factors of the SS polynomial a_0 (see (17) together with (14) and (15)), however, $\mathbf{S}'$ is not DLBM. Therefore, we apply the similarity transformation already mentioned to $\mathbf{S}'$ in the next section.

3.2. Second synthesis step

From state-space theory we know that it is possible to change the internal structure of a system by a regular transformation of its states without influencing the external behavior. This consideration can also be applied to our problem by showing that

$$\mathbf{S}''(z_1) = \begin{bmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(z_1) \end{bmatrix} \mathbf{S}'(z_1) \begin{bmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(z_1) \end{bmatrix}^{-1} \quad \begin{matrix} m \\ mn_2 \end{matrix} \quad (25)$$

$\begin{matrix} m & mn_2 \end{matrix}$

can be interpreted as the discrete scattering matrix of a one-dimensional system, N'' , which results in a two-dimensional network N , described by

the given matrix S , if the last mn_2 ports of N'' are terminated with z_2 -type reflectances. The matrix $T(z_1)$ is regular and rational in z_1 .

If we take into account (24) and (25) we have

$$S'' = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12}\mathbf{T}^{-1} \\ \mathbf{TH}_{21} & \mathbf{TH}_{22}\mathbf{T}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} \begin{matrix} m \\ mn_2 \end{matrix}, \quad (26)$$

$m \quad mn_2$

where

$$\mathbf{B}_{11} = \mathbf{H}_{11}, \quad (27a)$$

$$\mathbf{B}_{12} = \mathbf{H}_{12}\mathbf{T}^{-1}, \quad (27b)$$

$$\mathbf{B}_{21} = \mathbf{TH}_{21}, \quad (27c)$$

$$\mathbf{B}_{22} = \mathbf{TH}_{22}\mathbf{T}^{-1}. \quad (27d)$$

Using similar arguments as for the matrix S' and the network N' , the matrix S'' and the corresponding system N'' , respectively, will fulfill the above-defined condition if $\mathbf{B}_{11} + \mathbf{B}_{12}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{B}_{22})\mathbf{B}_{21} = S$. To show this, make use of (27) and (23):

$$\begin{aligned} \mathbf{B}_{11} + \mathbf{B}_{12}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{B}_{22})^{-1}\mathbf{B}_{21} &= \mathbf{H}_{11} + \mathbf{H}_{12}\mathbf{T}^{-1}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{TH}_{22}\mathbf{T}^{-1})^{-1}\mathbf{TH}_{21} \\ &= \mathbf{H}_{11} + \mathbf{H}_{12}(z_2^{-1}\mathbf{1}_{mn_2} - \mathbf{H}_{22})^{-1}\mathbf{H}_{21} = S. \end{aligned}$$

Thus, S'' has the desired property for every regular rational matrix $T(z_1)$.

In the following we want to determine the transformation matrix T in such a way that

$$S'' \begin{pmatrix} \mathbf{1}_{m+r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tilde{S}'' = \begin{pmatrix} \mathbf{1}_{m+r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (28)$$

where $r \leq mn_2$. The advantage of such a special transformation will be illuminated later. In order to determine T in terms of the elements of S' , we replace S'' of equation (28) by the definition (26) and compare corresponding block-elements on the left- and right-hand side of the resulting equation, which leads to the following system of equations:

$$\mathbf{H}_{11}\tilde{\mathbf{H}}_{11} + \mathbf{H}_{12}\mathbf{T}^{-1} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}^{-1}\tilde{\mathbf{H}}_{12} = \mathbf{1}_m, \quad (29a)$$

$$\mathbf{TH}_{21}\tilde{\mathbf{H}}_{11} + \mathbf{TH}_{22}\mathbf{T}^{-1} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}^{-1}\tilde{\mathbf{H}}_{12} = \mathbf{0}, \quad (29b)$$

$$\mathbf{H}_{11}\tilde{\mathbf{H}}_{21}\tilde{\mathbf{T}} + \mathbf{H}_{12}\mathbf{T}^{-1} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}^{-1}\tilde{\mathbf{H}}_{22}\tilde{\mathbf{T}} = \mathbf{0}, \quad (29c)$$

$$\mathbf{TH}_{21}\tilde{\mathbf{H}}_{21}\tilde{\mathbf{T}} + \mathbf{TH}_{22}\mathbf{T}^{-1} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}^{-1}\tilde{\mathbf{H}}_{22}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (29d)$$

Equations (29b) and (29c) are equivalent, since they are the discrete para-conjugate of each other. For convenience of notation, we define the matrix

$$\mathbf{K} = \mathbf{T}^{-1} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{T}}^{-1}, \quad (30)$$

thus, we can write the system (29) as

$$\mathbf{H}_{11}\tilde{\mathbf{H}}_{11} + \mathbf{H}_{12}\mathbf{K}\tilde{\mathbf{H}}_{12} = \mathbf{1}_m, \quad (31a)$$

$$\mathbf{H}_{21}\tilde{\mathbf{H}}_{11} + \mathbf{H}_{22}\mathbf{K}\tilde{\mathbf{H}}_{12} = \mathbf{0}, \quad (31b)$$

$$\mathbf{H}_{21}\tilde{\mathbf{H}}_{21} + \mathbf{H}_{22}\mathbf{K}\tilde{\mathbf{H}}_{22} = \mathbf{K}. \quad (31c)$$

Now we have the task of finding a rational matrix $\mathbf{K}$ which is a solution of the system of equations (31) and which can be decomposed as required by (30).

Theorem 3. *The matrix*

$$\mathbf{K} = \begin{bmatrix} u_0 \mathbf{1}_m & u_1 \mathbf{1}_m & \cdots & u_{n_2-1} \mathbf{1}_m \\ & u_0 \mathbf{1}_m & \cdots & u_{n_2-2} \mathbf{1}_m \\ & & \ddots & u_{n_2-3} \mathbf{1}_m \\ \mathbf{0} & & & \vdots \\ & & & u_0 \mathbf{1}_m \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \mathbf{1}_m & & & \\ \tilde{u}_1 \mathbf{1}_m & \tilde{u}_0 \mathbf{1}_m & & \mathbf{0} \\ \tilde{u}_2 \mathbf{1}_m & \tilde{u}_1 \mathbf{1}_m & & \\ \vdots & \vdots & \ddots & \\ \tilde{u}_{n_2-1} \mathbf{1}_m & \tilde{u}_{n_2-2} \mathbf{1}_m & \cdots & \tilde{u}_0 \mathbf{1}_m \end{bmatrix} \\ - \begin{bmatrix} \mathbf{N}_0 & \mathbf{N}_1 & \cdots & \mathbf{N}_{n_2-1} \\ & \mathbf{N}_0 & \cdots & \mathbf{N}_{n_2-2} \\ & & \ddots & \mathbf{N}_{n_2-3} \\ \mathbf{0} & & & \vdots \\ & & & \mathbf{N}_0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{N}}_0 & & & \\ \tilde{\mathbf{N}}_1 & \tilde{\mathbf{N}}_0 & & \mathbf{0} \\ \tilde{\mathbf{N}}_2 & \tilde{\mathbf{N}}_1 & & \\ \vdots & \vdots & \ddots & \\ \tilde{\mathbf{N}}_{n_2-1} & \tilde{\mathbf{N}}_{n_2-2} & \cdots & \tilde{\mathbf{N}}_0 \end{bmatrix} \quad (32)$$

is a solution of the system of equations (31), where the quantities u_i and $\mathbf{N}_i$ are given by (14) and (15), respectively.

Before this theorem can be proved, some further definitions have to be introduced. Due to our assumption, $\mathbf{S}(z)$ is a DLBM, so we have $\mathbf{S}\tilde{\mathbf{S}} = \mathbf{1}_m$ (see Definition 2). If we make use of (8), (12), (14), and (15) we obtain

$$\mathbf{S}\tilde{\mathbf{S}} = \frac{\sum_{i=0}^{n_2} \mathbf{N}_i z_2^i \sum_{i=0}^{n_2} \tilde{\mathbf{N}}_i z_2^{n_2-i}}{\sum_{i=0}^{n_2} u_i z_2^i \sum_{i=0}^{n_2} \tilde{u}_i z_2^{n_2-i}} = \mathbf{1}_m,$$

i.e.,

$$\left(\sum_{i=0}^{n_2} \mathbf{N}_i z_2^i \right) \left(\sum_{i=0}^{n_2} \tilde{\mathbf{N}}_i z_2^{n_2-i} \right) = \left(\sum_{i=0}^{n_2} u_i \mathbf{1}_m z_2^i \right) \left(\sum_{i=0}^{n_2} \tilde{u}_i \mathbf{1}_m z_2^{n_2-i} \right).$$

After equating both sides of the above equation and comparing the coefficients of equal powers of z_2 , we have

$$\sum_{l=0}^{n_2-i} (u_l \tilde{u}_{l+i} \mathbf{1}_m - \mathbf{N}_l \tilde{\mathbf{N}}_{l+i}) = \mathbf{0}, \quad \forall i = 0, \dots, n_2, \quad (33)$$

$$\sum_{l=j}^{n_2} (u_l \tilde{u}_{l-j} \mathbf{1}_m - \mathbf{N}_l \tilde{\mathbf{N}}_{l-j}) = \mathbf{0}, \quad \forall j = 0, \dots, n_2. \quad (34)$$

Furthermore, the $(m \times m)$ -block-elements of the matrix $\mathbf{K}$, defined by (32), are designated as

$$\mathbf{K}_{ij} = \sum_{l=\max(0, j-i)}^{n_2-i} (u_l \tilde{u}_{l+i-j} \mathbf{1}_m - \mathbf{N}_l \tilde{\mathbf{N}}_{l+i-j}), \quad i, j = 1, \dots, n_2. \quad (35)$$

For further consideration it is useful to extend the relation given above to the case $i = 0$ or $j = 0$ by defining

$$\mathbf{R}_{ij} = \sum_{l=\max(0, j-i)}^{n_2-i} (u_l \tilde{u}_{l+i-j} \mathbf{1}_m - \mathbf{N}_l \tilde{\mathbf{N}}_{l+i-j}), \quad i, j = 0, 1, \dots, n_2. \quad (36)$$

Lemma 1. For the quantities $\mathbf{R}_{ij}$, defined by (36), we have

$$\mathbf{R}_{ij} = \begin{cases} \mathbf{K}_{ij} & \text{for } i, j = 1, 2, \dots, n_2, \\ \mathbf{0} & \text{for } i = 0, \quad j = 0, 1, \dots, n_2, \\ \mathbf{0} & \text{for } j = 0, \quad i = 0, 1, \dots, n_2. \end{cases} \quad (37)$$

Proof. That (37) holds for $i, j = 1, \dots, n_2$ can be seen by comparing (35) with (36). $\mathbf{R}_{0j} = \mathbf{0}$ can be proved by comparing (36) with (34) and $\mathbf{R}_{i0} = \mathbf{0}$ by comparing (36) with (33). $\square$

Furthermore, we need the relation

$$\mathbf{R}_{i-1, j-1} = \mathbf{R}_{ij} + u_{n_2-i+1} \tilde{u}_{n_2-j+1} \mathbf{1}_m - \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_{n_2-j+1}, \quad i, j = 1, \dots, n_2, \quad (38)$$

which can be checked by using (36) and (38) and taking into account

$$\mathbf{R}_{i-1, j-1} = \sum_{l=\max(0, j-i)}^{n_2-i+1} (u_l \tilde{u}_{l+i-j} \mathbf{1}_m - \mathbf{N}_l \tilde{\mathbf{N}}_{l+i-j}), \quad i, j = 1, \dots, n_2.$$

Proof of Theorem 3. The validity of Theorem 3 will be proved by putting (32) into (31) and comparing the left-hand side (l.h.s.) and right-hand side (r.h.s.) of the system of equations.

(i) *Equation (31a).* By using (17a), (17c), and the left-hand side of (31a), we obtain

$$\text{l.h.s.} = \mathbf{N}_0 \tilde{\mathbf{N}}_0 + \mathbf{K}_{n_2 n_2},$$

and with (35) and (14) we have

$$\text{l.h.s.} = \mathbf{N}_0 \tilde{\mathbf{N}}_0 + \mathbf{1}_m - \mathbf{N}_0 \tilde{\mathbf{N}}_0 = \mathbf{1}_m = \text{r.h.s.}$$

(ii) *Equation (31b)*. By using (17) and Lemma 1, we obtain, for the $(m \times m)$ -block-element of the position (i) of the left-hand side of (31b),

$$(\text{l.h.s.})_i = \mathbf{A}_{n_2-i+1} \tilde{\mathbf{N}}_0 + \mathbf{R}_{i-1, n_2} - u_{n_2-i+1} \mathbf{K}_{n_2 n_2}, \quad i = 1, \dots, n_2,$$

and with (16), (35), and (36) we have

$$\begin{aligned} (\text{l.h.s.})_i &= \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_0 - \mathbf{N}_0 \tilde{\mathbf{N}}_0 u_{n_2-i+1} + u_{n_2-i+1} \tilde{u}_0 \mathbf{1}_m - \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_0 \\ &\quad - u_{n_2-i+1} u_0 \tilde{u}_0 \mathbf{1}_m + u_{n_2-i+1} \mathbf{N}_0 \tilde{\mathbf{N}}_0, \quad i = 1, \dots, n_2, \end{aligned}$$

i.e.,

$$\text{l.h.s.} = \mathbf{0} = \text{r.h.s.},$$

since $u_0 = 1$.

(iii) *Equation (31c)*. By using (17) and Lemma 1, we obtain, for the $(m \times m)$ -block-element of the position (i, j) of the left-hand side of (31c),

$$\begin{aligned} (\text{l.h.s.})_{i,j} &= \mathbf{A}_{n_2-i+1} \tilde{\mathbf{A}}_{n_2-j+1} + \mathbf{R}_{i-1, j-1} - u_{n_2-i+1} \mathbf{R}_{n_2, j-1} - \mathbf{R}_{i-1, n_2} \tilde{u}_{n_2-j+1} \\ &\quad + u_{n_2-i+1} \tilde{u}_{n_2-j+1} \mathbf{R}_{n_2 n_2}, \quad i, j = 1, \dots, n_2, \end{aligned}$$

and with (16), (38), (36), and (14) we have

$$\begin{aligned} (\text{l.h.s.})_{i,j} &= \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_{n_2-j+1} - u_{n_2-i+1} \mathbf{N}_0 \tilde{\mathbf{N}}_{n_2-j+1} - \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_0 \tilde{u}_{n_2-j+1} \\ &\quad + u_{n_2-i+1} \tilde{u}_{n_2-j+1} \mathbf{N}_0 \tilde{\mathbf{N}}_0 + \mathbf{R}_{ij} + u_{n_2-i+1} \tilde{u}_{n_2-j+1} \mathbf{1}_m - \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_{n_2-j+1} \\ &\quad - u_{n_2-i+1} (\tilde{u}_{n_2-j+1} \mathbf{1}_m - \mathbf{N}_0 \tilde{\mathbf{N}}_{n_2-j+1}) - \tilde{u}_{n_2-j+1} (u_{n_2-i+1} \mathbf{1}_m - \mathbf{N}_{n_2-i+1} \tilde{\mathbf{N}}_0) \\ &\quad + u_{n_2-i+1} \tilde{u}_{n_2-j+1} (\mathbf{1}_m - \mathbf{N}_0 \tilde{\mathbf{N}}_0), \quad i, j = 1, \dots, n_2, \end{aligned}$$

i.e.,

$$(\text{l.h.s.})_{i,j} = \mathbf{R}_{ij} = \mathbf{K}_{ij} = (\text{r.h.s.})_{i,j}, \quad i, j = 1, \dots, n_2. \quad \square$$

In the next step we show that the matrix $\mathbf{K}$, given by (32), can be decomposed as required by (30). But first consider the eigenvalues of the matrix $\mathbf{H}_{22}$. From (18), A3 (see the Appendix), and (19) we conclude

$$\det(z_2^{-1} \mathbf{1}_{mn_2} - \mathbf{H}_{22}) = \det \mathbf{M}_{11} \det(\mathbf{M}_{22} - \mathbf{M}_{21} \mathbf{M}_{11}^{-1} \mathbf{M}_{12}) = z_2^{-(n_2-1)m} \det \mathbf{D},$$

and with (21) we have

$$\det(z_2^{-1} \mathbf{1}_{mn_2} - \mathbf{H}_{22}) = \left(\frac{\hat{g}(z)}{z_2^{n_2} a_0(z_1)} \right)^m,$$

i.e.,

$$\det(\lambda \mathbf{1}_{mn_2} - \mathbf{H}_{22}) = \left(\frac{\lambda^{n_2} \hat{g}(z_1, 1/\lambda)}{a_0(z_1)} \right)^m. \quad (39)$$

Since $\hat{g}(z_1, z_2)$ is an SS polynomial, there exists an infinite set $\Omega \subseteq T^1$ such that

$$\hat{g}(z_1, z_2) \neq 0 \quad \text{for } |z_2| \leq 1 \quad \text{and } z_1 \in \Omega, \quad (40)$$

where $T^1 \setminus \Omega$ has at most a finite number of elements (see Theorem 5 together with Corollary 1 of [14]), i.e.,

$$\det(\lambda \mathbf{I}_{mn_2} - \mathbf{H}_{22}(z_1)) \neq 0 \quad \text{for } |\lambda| \geq 1 \quad \text{and } z_1 \in \Omega, \quad (41)$$

which implies that every eigenvalue λ_i , $i = 1$ to mn_2 , of the matrix $\mathbf{H}_{22}(z_1)$ is located in the interior of the unit circle U^1 , if $z_1 \in \Omega$. Hence, due to A4, the matrix $\mathbf{K}$, defined by (32), is a *unique* solution of equation (31c) for every $z_1 \in \Omega$.

Theorem 4. *The matrix $\mathbf{K}(z_1)$, defined by (32), is nonnegative definite Hermitian for $z_1 \in T^1$, and $\mathbf{K} = \bar{\mathbf{K}}$.*

Proof. Since all the eigenvalues of $\mathbf{H}_{22}(z_1)$ are strictly smaller than one in modulus and (31c) can be interpreted as a Liapunov equation for $z_1 \in \Omega$, the matrix

$$\mathbf{K} = \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{I} - e^{j\varphi} \mathbf{H}_{22})^{-1} \mathbf{H}_{21} \mathbf{H}_{21}^* (\mathbf{I} - e^{-j\varphi} \mathbf{H}_{22}^*)^{-1} d\varphi \quad (42)$$

is a solution of (31c), which has been shown in Theorem 4 of [15], i.e., it is equal to the matrix defined in (32) for $z_1 \in \Omega$, since the solution is unique for these points. The so-called Neumann series

$$(\mathbf{I} - e^{j\varphi} \mathbf{H}_{22})^{-1} = \sum_{i=0}^{\infty} (e^{j\varphi} \mathbf{H}_{22})^i, \quad \text{for } z_1 \in \Omega, \quad (43)$$

is convergent since all the eigenvalues of $e^{j\varphi} \mathbf{H}_{22}$ are in the interior of the unit-circle for $z_1 \in \Omega$ (see Section 20.2 of [16]). Thus, we obtain from (42) and (43)

$$\mathbf{K} = \frac{1}{2\pi} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \mathbf{H}_{22}^i \mathbf{H}_{21} \mathbf{H}_{21}^* \mathbf{H}_{22}^{*k} \int_0^{2\pi} e^{j(i-k)\varphi} d\varphi, \quad (44)$$

$$\mathbf{K} = \sum_{i=0}^{\infty} \mathbf{H}_{22}^i \mathbf{H}_{21} \mathbf{H}_{21}^* \mathbf{H}_{22}^{*i} \quad \text{for } z_1 \in \Omega.$$

$\mathbf{K}(z_1)$ is obviously nonnegative definite Hermitian for $z_1 \in \Omega$, i.e.,

$$y(z_1) = \mathbf{x}^* \mathbf{K}(z_1) \mathbf{x} \geq 0$$

for an arbitrary, constant column vector $\mathbf{x}$ and $z_1 \in \Omega$. On the other hand, $y(z_1)$ is a continuous function of z_1 , hence, it cannot be negative for the isolated points $z_1 \in T^1 \setminus \Omega$ exclusively, i.e.,

$$y(z_1) \geq 0 \quad \text{for } z_1 \in T^1.$$

Thus, $\mathbf{K}$ is nonnegative definite Hermitian for $z_1 \in T^1$, as $\mathbf{x}$ may be chosen arbitrarily. Furthermore, $\mathbf{K}$ obeys the property $\mathbf{K} = \tilde{\mathbf{K}}$, which can be seen from (32). $\square$

Due to the properties shown in Theorem 4, the one-variable rational matrix $\mathbf{K}$ can be factorized as

$$\mathbf{K} = \underset{r}{\mathbf{D}^{-1}} \begin{pmatrix} \mathbf{G} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underset{r}{\mathbf{D}^{-1}} = \underset{r}{\mathbf{D}^{-1}} \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tilde{\mathbf{D}}^{-1}, \quad (45)$$

where $\mathbf{D}(z_1)$ is a unimodular polynomial matrix, $\mathbf{G}(z_1)$ and $\mathbf{G}(z_1)^{-1}$ are rational, regular $(r \times r)$ -matrices, which are both holomorphic for $|z_1| < 1$, and

$$r = \text{rank } \mathbf{K} \quad (46)$$

(see [17]–[21]). If we define

$$\mathbf{T}^{-1} = \mathbf{D}^{-1} \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \mathbf{G}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \mathbf{D}, \quad (47)$$

then $\mathbf{T}$ and $\mathbf{T}^{-1}$ are regular rational matrices, which are both holomorphic for $|z_1| < 1$, since $\mathbf{G}$ and $\mathbf{G}^{-1}$ have the same properties and $\mathbf{D}$ is unimodular. Finally, putting (47) into (45) shows that $\mathbf{K}$ admits the factorization required by (30).

Now, after we have proved the existence of the desired transformation matrix $\mathbf{T}$, we want to investigate the special properties of the matrix $\mathbf{S}''$, which is defined by (25) and (26), respectively. To do this we partition the block-elements of $\mathbf{S}''$ (see (26)) as

$$\mathbf{B}_{12} = \begin{pmatrix} \mathbf{B}_{12}^{(1)} & \mathbf{B}_{12}^{(2)} \end{pmatrix} \begin{matrix} m, \\ mn_2 - r \end{matrix} \quad (48a)$$

$$\mathbf{B}_{21} = \begin{pmatrix} \mathbf{B}_{21}^{(1)} \\ \mathbf{B}_{21}^{(2)} \end{pmatrix} \begin{matrix} r \\ mn_2 - r \end{matrix}, \quad (48b)$$

$$\mathbf{B}_{22} = \begin{pmatrix} \mathbf{B}_{22}^{(1)} & \mathbf{B}_{22}^{(2)} \\ \mathbf{B}_{22}^{(3)} & \mathbf{B}_{22}^{(4)} \end{pmatrix} \begin{matrix} r \\ mn_2 - r \end{matrix}, \quad (48c)$$

and define

$$\mathbf{S}''' = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12}^{(1)} \\ \mathbf{B}_{21}^{(1)} & \mathbf{B}_{22}^{(1)} \end{pmatrix} \begin{matrix} m \\ r \end{matrix}. \quad (49)$$

By using (48), (26), (49), and (28), we obtain

$$\mathbf{S}'''\tilde{\mathbf{S}}''' = \mathbf{1}_{m+r}, \quad (50)$$

$$(\mathbf{B}_{21}^{(2)} \quad \mathbf{B}_{22}^{(3)}) \begin{pmatrix} \tilde{\mathbf{B}}_{21}^{(2)} \\ \tilde{\mathbf{B}}_{22}^{(3)} \end{pmatrix} = \mathbf{0}, \quad (51)$$

i.e.,

$$(\mathbf{B}_{21}^{(2)} \quad \mathbf{B}_{22}^{(3)}) \equiv \mathbf{0}. \quad (52)$$

This last result can be used to reduce the number of required frequency-dependent building elements. Hitherto, we had to terminate the network N'' with mn_2 z_2 -type reflectances, however, if we put (48) and (52) into the relation $\mathbf{S} = \mathbf{B}_{11} + \mathbf{B}_{12}(z_2^{-1}\mathbf{1} - \mathbf{B}_{22})^{-1}\mathbf{B}_{21}$, we have

$$\begin{aligned} \mathbf{S} &= \mathbf{B}_{11} + (\mathbf{B}_{12}^{(1)} \quad \mathbf{B}_{12}^{(2)}) \left(z_2^{-1}\mathbf{1}_{mn_2} - \begin{pmatrix} \mathbf{B}_{22}^{(1)} & \mathbf{B}_{22}^{(2)} \\ \mathbf{0} & \mathbf{B}_{22}^{(4)} \end{pmatrix} \right)^{-1} \begin{pmatrix} \mathbf{B}_{21}^{(1)} \\ \mathbf{0} \end{pmatrix}, \\ \mathbf{S} &= \mathbf{B}_{11} + \mathbf{B}_{12}^{(1)}(z_2^{-1}\mathbf{1}_r - \mathbf{B}_{22}^{(1)})^{-1}\mathbf{B}_{21}^{(1)}, \end{aligned} \quad (53)$$

which implies that the matrix $\mathbf{S}'''$, defined by (49), can be interpreted as the discrete scattering matrix of an $(m+r)$ -port N''' , which results in a two-dimensional m -port N , described by the discrete scattering matrix $\mathbf{S}$, if the last r ports of N''' are terminated with z_2 -type reflectances. Thus, we only need r z_2 -type reflectances instead of mn_2 if we use N''' as a coupling network for the synthesis of the desired network N . Furthermore, due to (25) the matrix $\mathbf{S}''$ and its block-element $\mathbf{S}'''$ are rational in z_1 and holomorphic for $|z_1| < 1$, since this holds for $\mathbf{S}'$, $\mathbf{T}$, and $\mathbf{T}^{-1}$ too. With this in mind and together with (50) we can conclude from Theorem 1 that $\mathbf{S}'''$ is a DLBM. Thus, the synthesis problem is solved, since $\mathbf{S}'''(z_1)$ can be realized with known methods as the discrete scattering matrix of a 1-D lossless network N''' . If we terminate the last r ports of N''' with z_2 -type reflectances, we obviously obtain the desired 2-D lossless m -port N , with discrete scattering matrix $\mathbf{S}$.

The number of required frequency-dependent building elements depends on r and the degree of $\det \mathbf{S}'''(z_1)$. The last section is devoted to this problem.

3.3. The number of building elements

First define the matrices

$$\mathbf{C}(z_1) = (\mathbf{H}_{21}, \mathbf{H}_{22}\mathbf{H}_{21}, \dots, \mathbf{H}_{22}^{mn_2-1}\mathbf{H}_{21}), \quad (54)$$

$$\mathbf{Q}(z_1) = (\mathbf{H}_{12}^T, \mathbf{H}_{22}^T\mathbf{H}_{12}^T, \dots, \mathbf{H}_{22}^{Tmn_2-1}\mathbf{H}_{12}^T)^T. \quad (55)$$

Theorem 5.

$$r = \text{rank } \mathbf{K} \leq \text{rank } \mathbf{C}. \quad (56)$$

Proof. Let q be the normal rank of $C(z_1)$. By using elementary row operations, i.e., by premultiplication by a unimodular matrix, V , $C(z_1)$ can always be transformed to the following form [22, Chapter 6.3]:

$$VC = \begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix} \begin{matrix} q \\ mn_2 - q \end{matrix}, \quad (57)$$

where $\mathbf{X}$ may have nonzero elements. Due to the Caley-Hamilton theorem, the matrix $\mathbf{H}_{22}^i$ is a linear combination of the matrices $\mathbf{1}, \mathbf{H}_{22}, \mathbf{H}_{22}^2, \dots, \mathbf{H}_{22}^{mn_2-1}$ for $i \geq mn_2$. Thus, we always obtain, according to (54) and (57), the structure

$$V\mathbf{H}_{22}^i\mathbf{H}_{21} = \begin{bmatrix} \mathbf{X}_i \\ \mathbf{0} \end{bmatrix} \begin{matrix} q \\ mn_2 - q \end{matrix}, \quad i = 0, 1, 2, \dots, \quad (58)$$

where the entries of $\mathbf{X}_i$ are without interest. The structure (58) holds true even for the case $i \rightarrow \infty$ and $z_1 \in \Omega$, since we have $\lim_{i \rightarrow \infty} V\mathbf{H}_{22}^i\mathbf{H}_{21} = \mathbf{0}$, which is a consequence of the fact that all the eigenvalues of $\mathbf{H}_{22}$ are, for these points, elements of U^1 , which implies [23] $\lim_{i \rightarrow \infty} \mathbf{H}_{22}^i = \mathbf{0}$. By using the structure (58) we obtain from (44)

$$\begin{aligned} VKV^* &= \sum_{i=0}^{\infty} V\mathbf{H}_{22}^i\mathbf{H}_{21}\mathbf{H}_{21}^*\mathbf{H}_{22}^{*i}V^* \\ &= \sum_{i=0}^{\infty} \begin{pmatrix} \mathbf{X}_i\mathbf{X}_i^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{matrix} q \\ mn_2 - q \end{matrix} \quad \text{for } z_1 \in \Omega. \end{aligned}$$

Since the unimodular matrix V is always regular, we can conclude

$$\text{loc-rank } \mathbf{K}(z_1) \leq q \quad \text{for } z_1 \in \Omega. \quad (59)$$

Inequality (59) holds for an infinite number of points, hence, it must be valid for the normal rank too, i.e.,

$$\text{rank } \mathbf{K} \leq q = \text{rank } C. \quad \square$$

Theorem 6.

$$r = \text{rank } \mathbf{K} = n'_2, \quad (60)$$

where

$$n'_2 = \deg_2 g.$$

Proof. If we freeze the variable z_1 at a fixed point $z_{10} \in T^1$, then

$$\mathbf{W}(z_2) = \mathbf{S}(z_{10}, z_2) = \frac{\mathbf{P}(z_{10}, z_2)}{\hat{g}(z_{10}, z_2)}$$

is a one-variable DLBM, which follows from Theorem 1 in connection with the properties

- (i) $W(z_2)$ is rational in z_2 ,
- (ii) $W(z_2)$ is holomorphic for $|z_2| < 1$, since $\hat{g}(z_{10}, z_2) \neq 0$ for $|z_2| < 1$ and $z_{10} \in T^1$ [14, Theorem 5],
- (iii) $\mathbf{W}\tilde{\mathbf{W}} = \mathbf{S}(z_{10}, z_2)\tilde{\mathbf{S}}(z_{10}, z_2) = \mathbf{1}$ for $z_{10} \in T^1$.

Furthermore,

$$\det \mathbf{W} = \det \mathbf{S}(z_{10}, z_2) = \sigma \frac{g(z_{10}, z_2)}{\hat{g}(z_{10}, z_2)},$$

i.e.,

$$l = \deg \det \mathbf{W} \leq \deg_2 g(z_1, z_2) = n'_2. \quad (61)$$

Thus, the one-variable DLBM $W(z_2)$ can be realized as the discrete scattering matrix of a 1-D lossless m -port, which contains l frequency dependent elements, which is a consequence of classical results (see [24]–[26] in connection with the bilinear transformation (4)). By using gyrators and ideal transformers all frequency-dependent elements in this network can be realized as unit-capacitances, i.e., as z_2 -type reflectances. After extracting all l elements from this network, we obtain a *constant* coupling multiport, described by the scattering matrix

$$\mathbf{W}' = \begin{pmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{pmatrix} \begin{matrix} m \\ l \end{matrix}.$$

Applying the same considerations as at the beginning of Section 3.1, we have the following connection between $\mathbf{W}'$ and $\mathbf{W}$:

$$\mathbf{W} = \mathbf{W}_{11} + \mathbf{W}_{12}(z_2^{-1}\mathbf{1}_l - \mathbf{W}_{22})^{-1}\mathbf{W}_{21}. \quad (62)$$

The eigenvalues of the matrix $z_2\mathbf{W}_{22}$ are smaller than one in modulus if $|z_2| < 1/|\mu_{\max}|$, where $\mu_{\max}$ is the largest eigenvalue of $\mathbf{W}_{22}$. Thus, the Neumann series [16, Section 20.2]

$$(\mathbf{1}_l - z_2\mathbf{W}_{22})^{-1} = \sum_{i=0}^{\infty} (z_2\mathbf{W}_{22})^i$$

is convergent for these points, and can be substituted into (62):

$$\mathbf{S}(z_{10}, z_2) = \mathbf{W} = \mathbf{W}_{11} + \sum_{i=0}^{\infty} \mathbf{W}_{12}\mathbf{W}_{22}^i\mathbf{W}_{21}z_2^{i+1}, \quad \text{for } |z_2| < \frac{1}{|\mu_{\max}|}. \quad (63)$$

With the same arguments, we can expand the right-hand side of (5) in a

convergent power series as

$$\mathbf{S}(z_{10}, z_2) = \mathbf{H}_{11}(z_{10}) + \sum_{i=0}^{\infty} \mathbf{H}_{12}(z_{10}) \mathbf{H}_{22}^i(z_{10}) \mathbf{H}_{21}(z_{10}) z_2^{i+1},$$

$$\text{for } |z_2| < \frac{1}{|\lambda_{\max}|}, \quad (64)$$

where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{H}_{22}(z_{10})$. By comparing (63) with (64) we can easily see that the right-hand sides of both equations are identical for $|z_2| < \min(1/|\lambda_{\max}|, 1/|\mu_{\max}|)$, i.e., for a compact domain in the z_2 -plane. Thus, the coefficients of all powers of z_2 have to be equal, i.e.,

$$\mathbf{W}_{12} \mathbf{W}_{22}^i \mathbf{W}_{21} = \mathbf{H}_{12}(z_{10}) \mathbf{H}_{22}^i(z_{10}) \mathbf{H}_{21}(z_{10}), \quad \forall i = 0, 1, \dots \quad (65)$$

Let us define the matrices

$$\mathbf{C}' = (\mathbf{W}_{21}, \mathbf{W}_{22} \mathbf{W}_{21}, \dots, \mathbf{W}_{22}^{mn_2-1} \mathbf{W}_{21}), \quad (66a)$$

$$\mathbf{Q}' = (\mathbf{W}_{12}^T, \mathbf{W}_{22}^T \mathbf{W}_{12}^T, \dots, \mathbf{W}_{22}^{Tmn_2-1} \mathbf{W}_{12}^T)^T. \quad (66b)$$

With (54), (55), (66), and (65) we have

$$\mathbf{Q}(z_{10}) \mathbf{C}(z_{10}) = \mathbf{Q}' \mathbf{C}'.$$

Since the rank of $\mathbf{C}'$ does not exceed l , so it follows

$$\text{loc-rank } \mathbf{Q}(z_{10}) \mathbf{C}(z_{10}) = \text{rank } \mathbf{Q}' \mathbf{C}' \leq l. \quad (67)$$

Furthermore, if we put (17c) and (17d) into (55), we can see

$$\text{loc-rank } \mathbf{Q}(z_{10}) = mn_2. \quad (68)$$

Sylvester's inequality [22, A18] applied to (67) implies

$$\text{loc-rank } \mathbf{Q}(z_{10}) + \text{loc-rank } \mathbf{C}(z_{10}) - mn_2 \leq \text{loc-rank } \mathbf{Q}(z_{10}) \mathbf{C}(z_{10}) \leq l,$$

and with (68) and (61)

$$\text{loc-rank } \mathbf{C}(z_{10}) \leq n'_2.$$

Since this inequality is true for every $z_{10} \in T^1$, and the local rank of a rational one-variable matrix can differ from the normal rank for at most a finite number of points, we can conclude $\text{rank } \mathbf{C}(z_1) \leq n'_2$ and with (56) $r \leq n'_2$.

Assume $r < n'_2$, then we would be able to find, with the help of (49) and (53), a realization of $\mathbf{S}(z_{10}, z_2)$ which requires less than n'_2 frequency-dependent elements; but on the other hand, there exist points $z_{10} \in T^1$ for which the equality sign in (61) holds, i.e., every realization of $\mathbf{W}(z_2) = \mathbf{S}(z_{10}, z_2)$ requires at least $l = n'_2$ frequency-dependent elements [26], which contradicts our assumption, thus $r = n'_2$. $\square$

Theorem 7. *If we write $g(z_1, z_2)$ as*

$$g(z_1, z_2) = g_0(z_1) + g_1(z_1)z_2 + \cdots + g_{n'_2}(z_1)z_2^{n'_2},$$

then $\deg g_{n'_2}(z_1) = \deg g = n'_1$, and $\hat{g}_{n'_2} = a_0$.

Proof. According to the definition of the reciprocal conjugate of a polynomial we have

$$\hat{g} = \tilde{g}z_1^{n'_1}z_2^{n'_2} = \tilde{g}_{n'_1}z_1^{n'_1} + \tilde{g}_{n'_1-1}z_1^{n'_1-1}z_2 + \cdots + \tilde{g}_0z_1^{n'_1}z_2^{n'_2},$$

and with (12b)

$$\hat{g}(z_1, 0) = a_0(z_1) = \tilde{g}_{n'_1}(z_1)z_1^{n'_1}.$$

Assume $\deg g_{n'_2}(z_1) < n'_1$, then $a_0(z_1) = \tilde{g}_{n'_1}z_1^{n'_1}$ would have a zero at $z_1 = 0$, which contradicts (13). Thus, we have $\deg g_{n'_2}(z_1) = n'_1$, which implies $a_0 = \hat{g}_{n'_2}$. $\square$

Theorem 8.

$$\det \mathbf{S}'''(z_1) = (-1)^{n'_2} \sigma \frac{g_{n'_2}(z_1)}{\hat{g}_{n'_2}(z_1)}.$$

Proof. From (53) and (60) we have

$$\tilde{\mathbf{S}} = \tilde{\mathbf{B}}_{11} + \tilde{\mathbf{B}}_{21}^{(1)}(z_2 \mathbf{1}_{n'_2} - \tilde{\mathbf{B}}_{22}^{(1)})^{-1} \tilde{\mathbf{B}}_{12}^{(1)}$$

and by using A2 (see the Appendix), the discrete paraunitarity of $\mathbf{S}$, and (7), we obtain

$$\det \tilde{\mathbf{S}} = (-1)^{n'_2} \frac{\det \begin{bmatrix} \tilde{\mathbf{B}}_{11} & \tilde{\mathbf{B}}_{21}^{(1)} \\ \tilde{\mathbf{B}}_{12}^{(1)} & (\tilde{\mathbf{B}}_{22}^{(1)} - z_2 \mathbf{1}_{n'_2}) \end{bmatrix}}{\det(z_2 \mathbf{1}_{n'_2} - \tilde{\mathbf{B}}_{22}^{(1)})} = \det \mathbf{S}^{-1} = \frac{1}{\det \mathbf{S}} = \frac{\hat{g}}{\sigma g}. \quad (69)$$

Hence, since $\hat{g}$ and g are relatively prime, and $g/g_{n'_2}$ and $\det(z_2 \mathbf{1}_{n'_2} - \tilde{\mathbf{B}}_{22}^{(1)})$ are two polynomials in z_2 of the same degree n'_2 with rational coefficients in z_1 , where the coefficient of the highest power of z_2 is equal to one, we can conclude from (69)

$$\frac{g}{g_{n'_2}} = \det(z_2 \mathbf{1}_{n'_2} - \tilde{\mathbf{B}}_{22}^{(1)}). \quad (70)$$

By putting (70) into (69) and taking into account (49) we have

$$\det \left(\tilde{\mathbf{S}}''' - z_2 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n'_2} \end{pmatrix} \right) = (-1)^{n'_2} \frac{\hat{g}}{\sigma g_{n'_2}},$$

and with (50) and for $z_2 = 0$

$$\det \tilde{S}''' = (-1)^{n'_2} \frac{a_0}{\sigma g_{n'_2}} = \det S'''^{-1} = \frac{1}{\det S'''},$$

and with Theorem 7

$$\det S''' = (-1)^{n'_2} \sigma \frac{g_{n'_2}}{\hat{g}_{n'_2}}. \quad \square$$

Due to Theorems 7 and 8, the 1-D coupling network N''' , described by S''' , can be realized with n'_1 reactive elements. After terminating the last n'_2 ports of N''' with z_2 -type reflectances (see Theorem 6 in connection with (53)) we obtain the desired 2-D network N , described by S . Thus, we have found a minimal realization. To illustrate this, suppose there exists a realization of S with less than n'_2 z_2 -dependent elements. If we replace the z_1 -type reflectances in this network by *constant* one-ports with reflectance z_{10} , where $z_{10} \in T^1$ is chosen in such a way that $\deg_{z_2} g(z_{10}, z_2) = n'_2$, we would have a realization of $W(z_2) = S(z_{10}, z_2)$. But in the proof to Theorem 6 we have seen that every realization of $W(z_2)$ requires at least n'_2 z_2 -dependent elements, which contradicts the assumption. With similar arguments it can be shown that n'_1 is the minimal number of z_1 -dependent elements in any realization of S .

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Appendix

A1. If M_{11} and M_{22} are square matrices and M_{11} and $D = (M_{22} - M_{21}M_{11}^{-1}M_{12})$ are regular, then we have [16, Section 22.4]

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1} = \begin{bmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}D^{-1}M_{21}M_{11}^{-1} & -M_{11}^{-1}M_{12}D^{-1} \\ -D^{-1}M_{21}M_{11}^{-1} & D^{-1} \end{bmatrix}.$$

A2. If M_{11} and M_{22} are square matrices and M_{22} is regular, then we have [16, Section 22.4]

$$\det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \det M_{22} \det(M_{11} - M_{12}M_{22}^{-1}M_{21}).$$

A3. If M_{11} and M_{22} are square matrices and M_{11} is regular, then we have [16, Section 22.4]

$$\det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \det M_{11} \det(M_{22} - M_{21}M_{11}^{-1}M_{12}).$$

A4. If every eigenvalue λ_i of $\mathbf{H}$ is smaller than one in modulus, i.e., $|\lambda_i| < 1$, then the equation

$$\mathbf{H}\mathbf{K}\mathbf{H}^* - \mathbf{K} = -\mathbf{F}\mathbf{F}^*$$

has a unique solution $\mathbf{K}$, where $\mathbf{F}$ is an arbitrary matrix.

Proof. Simple matrix calculations show that

$$\mathbf{B}\mathbf{A} + \mathbf{A}\mathbf{B}^* = \mathbf{H}\mathbf{K}\mathbf{H}^* - \mathbf{K},$$

where $\mathbf{B} = \frac{1}{2}(\mathbf{H} - \mathbf{I})(\mathbf{H} + \mathbf{I})^{-1}$, $\mathbf{A} = (\mathbf{H} + \mathbf{I})\mathbf{K}(\mathbf{H}^* + \mathbf{I})$. Due to Section 20.1 of [16] the eigenvalues μ_i of $\mathbf{B}$ are given as $\mu_i = \frac{1}{2}((\lambda_i - 1)/(\lambda_i + 1))$, i.e., $\text{Re } \mu_i < 0$, since $|\lambda_i| < 1$. Hence, $\mathbf{B}$ and $-\mathbf{B}^*$ have no common eigenvalue, which implies [27, Section 8.3] that $\mathbf{B}\mathbf{A} + \mathbf{A}\mathbf{B}^* = -\mathbf{F}\mathbf{F}^*$ has a unique solution $\mathbf{A}$, and thus, $\mathbf{K} = (\mathbf{H} + \mathbf{I})^{-1}\mathbf{A}(\mathbf{H}^* + \mathbf{I})^{-1}$ is a unique solution of $\mathbf{H}\mathbf{K}\mathbf{H}^* - \mathbf{K} = -\mathbf{F}\mathbf{F}^*$, since $(\mathbf{H} + \mathbf{I})$ is regular for $|\lambda_i| < 1$. $\square$

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